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THE THEORY OF FORM LOGIC

So were it not for the mathematician's biassed interest he would invent a symbolism which was completely symmetrical as regards individuals and qualities.

Frank P. Ramsey, *Universals*

Abstract. We investigate a construction schema for first-order logical systems, called "form logic". Form logic allows us to overcome the dualistic commitment of predicate logic to individual constants and predicates. Dualism is replaced by a pluralism of terms of different "logical forms". Individual form-logical systems are generated by the determination of a range of logical forms and of the formbased syntax rules for combining terms into formulas. We develop a generic syntax and semantics for such systems and provide a completeness proof for them. To illustrate the idea of form logic, and the possibilities it facilitates, we discuss three particular systems, one of which is the form-logical reconstruction of standard first-order predicate logic.

Keywords: form logic, particular-universal distinction, three-valued logic, logical form, Wittgenstein.

1. Introduction

First-order predicate logic (PL) presupposes the classification of the non-logical terms into exactly two categories, individual and predicate terms, and gives a formation rule for atomic propositions based on those categories: Atomic formulas are formed by combining an n-place predicate and n individual terms. The commitment to exactly two types of

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non-logical terms comes with specific restrictions on the syntax rules for atomic and quantified formulas. Given intuitive formalization rules, "Black is beautiful", "Rolling over the floor is fun" and "Russell has three valuable properties" cannot be rendered in first-order predicate logic. On the other hand, PL permits formalization of arguably meaningless sentences like "The number 1 is rolling over the floor" and "Colorless green ideas sleep furiously".

Why should we accept the particular syntax of PL? Why should we not classify the non-logical terms into, say, 42 different syntactical categories and formulate rules which take account of this classification? Such a syntax may result in a more restrictive system than predicate logic, perhaps ruling out the two arguably meaningless sentences from above. On the other extreme, why not shoehorn all terms into a single syntactical category and advocate a more liberal system in which any n-tuple of terms is a well-formed formula? Such a system would be able to represent not only the notorious "Colorless green ideas sleep furiously", but also such syntactic oddities as "The number 3 is greater than" and "Russell Frege". Antecedently to any further argumentation, the formation rule for atomic formulas in PL appears arbitrary and no better off than possible alternatives.

For Gottlob Frege, the chosen syntax rules for atomic propositions are by no means due to arbitrary choice, but grounded in the very nature of sentences:

> We shall not stop at equations and inequalities. The linguistic form of equations is a statement. [...] Statements in general, just like equations or inequalities [...], can be imagined to be split up into two parts; one complete in itself, and the other in need of supplementation, or "unsaturated". Frege 1960/1891, p. 31

In the sentence "Caesar conquered Gaul", Frege claims, "Caesar" is *complete*, while "conquered Gaul" is *incomplete*: A sentence contains two types of expressions, an incomplete concept-expression, standing for an (incomplete) concept, and complete individual-expressions, standing for (complete) objects. A complete sense, a "thought", surfaces only where the empty space of the concept is saturated by a (complete) object. The dualistic syntax of predicate logic then arguably reflects the distinction between incomplete and complete sub-propositional terms. As the quote suggests and context confirms, the Fregean claim that all sentences contain an incomplete predicate and complete individual expressions is prompted by his analysis of identity. In identity statements, Frege holds, there is an incomplete expression, "=" or "is identical to", flanked by two complete expressions, singular terms. Frege's conception of identity therefore seems to be the motivational basis for predicate-logical dualism.

Ludwig Wittgenstein criticized this view on identity. He famously denies that there is an objectual identity relation at all: "That identity is not a relation between objects is obvious" (1922, §5.5301). He elaborates: "Roughly speaking: to say of *two* things that they are identical is nonsense, and to say of *one* thing that it is identical with itself is to say nothing" (Wittgenstein 1922, §5.5303).[1] Wittgenstein concludes that a proper symbolism must dispense with the identity sign. He proposes the alternative option of showing identity and difference by means of notational choice: "Identity of the object I express by identity of the sign and not by means of a sign of identity. Difference of the objects by difference of the signs" (Wittgenstein 1922, §5.53). In subsequent passages (§§5.531–5.5321), Wittgenstein attempts to undermine indispensability arguments by providing an identity-free notation of first-order formulas involving the identity sign. Only much later Jaakko Hintikka (1956) and, more recently, Kai F. Wehmeier (e.g., 2004) succeeded in formulating first-order predicate-logical systems representing a *Wittgensteinian predicate logic*, which dispense with the identity sign while maintaining the expressive power of predicate logic with identity.[2]

Wittgenstein's criticism of identity has found some response amongst philosophers and logicians, but a more basic discontent, pertaining to Frege's analysis of propositions in general, has been largely ignored or at least insufficiently adverted to.[3] If there is no objectual identity relation, an identity statement does not express a Fregean thought (cf. Wittgenstein 1922, §§6.2–6.21) and *a fortiori* is not amenable to an analysis in

[1] Wittgenstein's criticism clearly echoes an objection which already Bertrand Russell considered in his *Principles of Mathematics*, §64: "The question whether identity is or is not a relation, and even whether there is such a concept at all, is not easy to answer. For it may be said, identity cannot be a relation, since, where it is truly asserted, we have only one term, whereas two terms are required for a relation. And indeed identity, an objector may urge, cannot be anything at all: two terms plainly are not identical, and one term cannot be, for what is it identical with?" (1903, p. 63).

[2] In a recent paper (forthcoming), Wehmeier even claims that we can dispense with any form of objectual identity in ordinary language, by giving a different explanation of apparent identity statements.

[3] A notable exception is, of course, Ramsey (1925).

terms of the distinction between complete and incomplete expressions. Identity statements therefore cannot provide the model for analyzing atomic propositions. But in as much as this analysis is unsupported, so is the Fregean view that the syntax of PL reflects the logical syntax of ordinary language.

In contrast to Frege, Wittgenstein understands atomic sentences not as combinations of predicate and individual terms: "The atomic proposition consists of names. It is a connexion, a concatenation, of names" (Wittgenstein 1922, §4.22). By calling all non-logical constants 'names', Wittgenstein does not, however, reject the idea of incomplete terms. On a plausible interpretation, he suggests that *all* names are incomplete, referring to objects (Wittgenstein 1922, §§2.13–2.131, §3.22) which themselves are incomplete constituents of states of affairs (Wittgenstein 1922, §2.01). The Wittgensteinian alternative to Frege is therefore not the rejection of incompleteness, but its generalization. *Any* sub-propositional term is incomplete.[4]

This opens up new syntactical possibilities. While Frege claims that all atomic propositions essentially possess the same syntactical structure (n-place predicate, n individual terms), Wittgenstein thinks that they can be of very different forms. Restricting himself to the simplified model of sentences containing a subject and a one-place predicate, he says:

> The fact that we use subject-predicate propositions is only a matter of our notation. The subject-predicate form does not in itself amount to a logical form and is the way of expressing countless fundamentally different logical forms [...]. One difficulty in the Fregean theory is the generality of the words 'concept' and 'object'. For even if you can count tables and tones and vibrations and thoughts, it is difficult to bracket them all together. Concept and object: but that is subject and predicate. And we have just said that there is not just one logical form which is *the* subject-predicate form. Wittgenstein 1980/1964, §93

According to Wittgenstein, the subject-predicate form of ordinary-language propositions does not determine a single logical form and *a fortiori* not *the* logical form of a proposition. "Russell is wise" and "2 is even" do not possess the same logical form, for this would mean that

[4] For this interpretation see, e.g., Eric Stenius 1976, p. 80: "According to what I believe to be the theory which Wittgenstein tends to embrace in his *Tractatus all* 'things' are what Frege calls 'unsaturated'." Ramsey (1925, pp. 403, 408), influenced by Wittgenstein, also characterizes objects as incomplete. For further discussion, see Freitag 2009, pp. 20 ff.

we could swap the predicates *salva congruitate*. Exchange of the 'predicates', however, would result in meaningless strings of signs and therefore violate the conditions on a proper symbolism. In a Wittgensteinian *Begriffsschrift*, as we might call a system which respects Wittgenstein's demands, all well-formed propositions are meaningful, i.e., express possible states of affairs.

Wittgensteinian "names" (encompassing, recall, ordinary 'predicates') are all incomplete, but they are not all incomplete in the same way. A name is classified with respect to its "logical form", determining its possibility of combining, together with other names, into atomic propositions representing possible atomic states of affairs. "Russell" and "the number 2" fall into different categories since they have different logical forms. Where Frege classifies non-logical terms with respect to their being incomplete or not, Wittgenstein classifies them according to the specific *ways* in which they are incomplete. This change of perspective generates the possibility of replacing the Fregean dualism with a pluralism of terms of different forms and hence the possibility of an alternative syntax for atomic propositions.

Our aim is not to settle the dispute on incompleteness between Frege and Wittgenstein, nor even to decide whether it is an appropriate dispute in the present context. The mere possibility of disagreement concerning the question of the logical form of atomic propositions suggests that the PL-syntax of atomic formulas is not indisputable. Alternative logical systems are conceivable. Pending further argumentation, PL is only one of many possible systems, the choice of and justification for which will depend on our needs and persuasions.

The aim of this paper is the investigation of a construction schema for logical systems which does justice to the possibility of choice and therefore allows the implementation of arbitrary syntactic classifications of the non-logical terms. Besides a Fregean predicate logic and a Wittgensteinian *Begriffsschrift*, the schema should also permit formal languages in which there are, say, exactly 42 categories of non-logical terms, and systems with no category distinctions at all. In deference to Wittgenstein, we call this construction schema *form logic* and the individual systems *form-logical* systems.

Here is an informal synopsis of form-logic. We first introduce the syntax. While in first-order predicate logic atomic formulas are defined by means of predicate terms and individual terms, i.e., have the form $\mathrm{P}^n(\tau_1 \ldots \tau_n)$ where n is the arity of the predicate P and the τ_i's are individual terms, in form-logical systems atomic formulas are sequences of terms *simpliciter*, i.e., have the form $\tau_1 \ldots \tau_n$. However, not any sequence corresponds to a well-formed formula. In each form-logical system S the terms (i.e. the constants and variables) of the alphabet are partitioned into so-called "term-form classes", $\mathfrak{T}_i$. S is equipped with a class $\mathfrak{S}^{\mathfrak{T}}$ of finite sequences of term-form classes. A well-formed sequence of terms respects the forms of the occurring terms: only sequences of terms whose respective sequence of term-form classes is in $\mathfrak{S}^{\mathfrak{T}}$ are well-formed. Predicate logic is then merely a special form-logical system.[5]

We present a semantics for form-logic, in which we assign to any term an object of the domain. Truth is then defined with respect to an extension and an anti-extension, which are sets of tuples of objects. Intuitively, the extension is the set of possible (according to the system) and obtaining atomic states of affairs, and the anti-extension is the set of possible (according to the system) but non-obtaining atomic states of affairs. It will be seen that many form-logical systems, not the form-logical reconstruction of PL, however, are three-valued.

Once the construction schema has been introduced, we discuss three sample form-logical systems. First, we present a simple system without any category distinctions. Then we discuss the form-logical reconstruction of predicate logic. Finally, we illustrate the idea of a Wittgensteinian *Begriffsschrift* with respect to a toy universe.

Certain meta-logical results are available. A proof for the strong completeness of all form-logical systems will be given in the appendix.

2. The Formal System of Form Logic

In this section, we present the construction schema for form-logical systems.

Alphabet.

(i) Terms: constants: $a_1, a_2, \ldots$; variables: $x_1, x_2, \ldots$

[5] Since form logic is not distinctive with respect to issues surrounding identity, we will construct the system without identity. Systems with an identity sign can be easily formulated. Alternatively, it is also possible to incorporate the Hintikka-Wehmeierstyle of dispensing with identity, while obtaining the same expressive power as in a system with identity.

(ii) Logical constants: propositional connectives: $\vee$, $\neg$;[6] quantifier: $\forall$.[7]
(iii) Punctuation: parentheses: (,)

Syntax. To every form-logical system S there belongs a partition of the set T of terms. The equivalence classes introduced by the partition are called *term-form classes*. The partition must be chosen such that if there are variables in a term-form class, then there are countably infinitely many variables in the respective term-form class. ${}_S\mathfrak{T}^*$ is the set of the term-form classes of S. The function ${}_Sf^{\mathrm{T}} : \mathrm{T} \to {}_S\mathfrak{T}^*$ maps each term to its term-form class. To every form-logical system S there also belongs a set of finite sequences of term-form classes, ${}_S\mathfrak{S}^{\mathfrak{T}}$.

The syntax rule for *atomic* formulas is the following:[8]

$\tau_1 \ldots \tau_n$ is an atomic well-formed formula of the form-logical system S iff $f^{\mathrm{T}}(\tau_1) \ldots f^{\mathrm{T}}(\tau_n) \in {}_S\mathfrak{S}^{\mathfrak{T}}$.

The syntax rules for *molecular* formulas are those of (PL). The set of well-formed formulas of a form-logical system S is thus uniquely determined by ${}_Sf^{\mathrm{T}}$ and ${}_S\mathfrak{S}^{\mathfrak{T}}$, onto which form logic as such does not impose any substantial restrictions. Form logic therefore has the syntactical variability which we have demanded; it allows for a great number of different form-logical systems.[9]

Semantics. A *structure* A is defined as a pair $\langle D, i\rangle$ consisting of a non-empty set D, called the *domain* of A, and a function i from the set of constants into the domain. A *variable assignment* β is a function from the set of variables into a subset D^V of the domain. To every system S belongs an *interpretation* I, which is a pair $\langle A, \beta\rangle$ consisting of a structure $A = \langle D, i\rangle$ and a variable assignment β. We define:

(i) $I(a_j) = i(a_j)$ for all constants a_j
(ii) $I(x_j) = \beta(x_j)$ for all variables x_j,
(iii) $I(\tau_1 \ldots \tau_n) = (I(\tau_1), \ldots, I(\tau_n))$ for all atomic formulas $\tau_1 \ldots \tau_n$.

If $f^{\mathrm{T}}(a_j) \neq f^{\mathrm{T}}(a_k)$ (for arbitrary $j, k \in \mathbb{N}$), then i must be chosen such that $i(a_j) \neq i(a_k)$.

[6] The other propositional connectives ("$\wedge$", "$\rightarrow$", "$\leftrightarrow$") can be defined via these connectives in the usual manner.

[7] The existential quantifier "$\exists x$" can be introduced as an abbreviation for "$\neg\forall x\neg$".

[8] Here and in what follows, we use τ_1, …, τ_n as meta-language parameters for terms, and φ, χ, ψ for formulas.

[9] In the following, we avoid indexing with S wherever context disambiguates.

Note that, semantically, we treat all non-logical terms alike; in particular, the function i assigns to all non-logical constants elements of the domain. This formally encodes Wittgenstein's view that all sub-propositional signs are names and that names stand for, or represent, simple objects.

Synontix. To define truth in a model, we first introduce a classification of the elements of the domain into object-form classes. As the syntax states the rules for the combination of the signs, the *synontix* states the rules for the combinations of the ontic elements, i.e., the elements of the domain.

We first choose an enumeration of the term-form classes: $\mathfrak{T}_1, \mathfrak{T}_2, \ldots$. Then, we choose a partition of the domain D into object-form classes $\mathfrak{O}_j$ as follows: If $a_j \in \mathfrak{T}_k$ then $i(a_j) \in \mathfrak{O}_k$, for all $\mathfrak{T}_k \in \mathfrak{T}^*$ and $j \in \mathbb{N}$. Form classes with an identical index are said to *correspond* to each other (i.e. $\mathfrak{T}_k$ and $\mathfrak{O}_k$ are corresponding form-classes for all $k \in \mathbb{N}$). Objects without a name are classified in arbitrary object-form classes corresponding to term-form classes with variables.[10] (Note that we allow object-form classes that contain only unnamed objects. Such an object-form class then corresponds to a term-form class that only contains variables.)

$\mathfrak{O}^*$ is the set of all object-form classes. The function $f^D : D \to \mathfrak{O}^*$ maps every object of the domain to its object-form class.

To every form-logical system S there belongs a set ${}_S\mathfrak{S}^{\mathfrak{O}}$ of finite sequences of object-form classes. We choose ${}_S\mathfrak{S}^{\mathfrak{O}}$ such that for every sequence of term-form classes in ${}_S\mathfrak{S}^{\mathfrak{T}}$ there is a corresponding sequence of object-form classes in ${}_S\mathfrak{S}^{\mathfrak{O}}$, where a sequence of object-form classes and a sequence of term-form classes are called *corresponding* if and only if there are corresponding form classes at all places of the sequences. No other sequences of object-form classes appear in ${}_S\mathfrak{S}^{\mathfrak{O}}$.

This construction guarantees that every well-formed atomic formula expresses a "legitimate" (atomic) state of affairs, i.e., a state of affairs such that its respective sequence of object-form classes is in ${}_S\mathfrak{S}^{\mathfrak{O}}$.

Truth in a model. We choose a set E of sequences of elements of the domain, called the *extension*, and a set E^-, called the *anti-extension*,

[10] The correspondence between term-form classes and object-form classes implements Wittgenstein's idea of isomorphy, a cornerstone of his picture theory of meaning. A picture, and hence a sentence, and the pictured reality must have in common the "logical form" (Wittgenstein 1922, §2.161–2.2).

such that

$$E \cap E^- = \varnothing \text{ and } E \cup E^- = \{a_1, \ldots, a_n \mid f^D(a_1) \ldots f^D(a_n) \in \mathfrak{S}^{\mathfrak{D}}\}.$$

Intuitively, $E \cup E^-$ is the set of legitimate atomic states of affairs of a logical system S. The extension E is then the set of *obtaining* atomic states of affairs and the anti-extension E^- is the set of *non-obtaining* states of affairs. A *model* I_{E,E^-} is a triple consisting of an interpretation, an extension and an anti-extension: $I_{E,E^-} = \langle I, E, E^- \rangle$.

Form-logic permits the construction of three-valued systems. We will explain the reasons for this in the next paragraph after we have given the exact truth-conditions for molecular formulas. We indicate the third truth-value with "$*$" and use a strong Kleene semantics for molecular formulas:

$$\begin{aligned}
I_{E,E^-}(\tau_1 \ldots \tau_n) = \text{TRUE} \quad &:\Leftrightarrow \quad I(\tau_1, \ldots, \tau_n) \in E,\\
I_{E,E^-}(\tau_1 \ldots \tau_n) = \text{FALSE} \quad &:\Leftrightarrow \quad I(\tau_1, \ldots, \tau_n) \in E^-,\\
I_{E,E^-}(\neg\varphi) = \text{TRUE} \quad &:\Leftrightarrow \quad I_{E,E^-}(\varphi) = \text{FALSE},\\
I_{E,E^-}(\neg\varphi) = \text{FALSE} \quad &:\Leftrightarrow \quad I_{E,E^-}(\varphi) = \text{TRUE},
\end{aligned}$$

$$\begin{aligned}
I_{E,E^-}(\varphi \vee \psi) = \text{TRUE} \quad &:\Leftrightarrow \quad I_{E,E^-}(\varphi) = \text{TRUE or } I_{E,E^-}(\psi) = \text{TRUE},\\
I_{E,E^-}(\varphi \vee \psi) = \text{FALSE} \quad &:\Leftrightarrow \quad I_{E,E^-}(\varphi) = \text{FALSE and } I_{E,E^-}(\psi) = \text{FALSE},\\
I_{E,E^-}(\forall x\, \varphi) = \text{TRUE} \quad &:\Leftrightarrow \quad I_{E,E^-}a/x(\varphi) = \text{TRUE for all a } \in D^V,^{11}\\
I_{E,E^-}(\forall x\, \varphi) = \text{FALSE} \quad &:\Leftrightarrow \quad I_{E,E^-}a/x(\varphi) = \text{FALSE for at least one}\\
&\qquad\; a \in D^V,
\end{aligned}$$

$$I_{E,E^-}(\varphi) = \; * \quad :\Leftrightarrow \quad \text{neither } I_{E,E^-}(\varphi) = \text{TRUE nor } I_{E,E^-}(\varphi) = \text{FALSE}.$$

Form-logical semantics differs from classical model-theoretical semantics in that the truth-values of the formulas do not supervene on the given interpretation.

Paracompleteness. As already noted, there are form-logical systems which are three-valued, i.e., for which $\varphi \vee \neg\varphi$ is not a theorem. Such systems are paracomplete.

Closed atomic formulas are either true of false. Open and quantified formulas can take on the third truth-value, because it can happen that one has to take into account non-legitimate states of affairs to evaluate

[11] As usual, $\beta a/x$ (with β being an assignment and $a \in D^V$) is defined as follows: $\beta a/x(y) = \beta(y)$ if $y \neq x$ and $\beta a/x(y) = a$ if $y = x$. Furthermore, Ia/x (with $a \in D^V$ and $I = (A, \beta)$) is defined as $(A, \beta a/x)$.

them. For illustration, consider the following example: $D^V = \{a_1, a_2\}$, $a_1 \in \mathfrak{O}_1$, $a_2 \in \mathfrak{O}_2$, $\mathfrak{S}^{\mathfrak{O}} = \{\mathfrak{O}_1\mathfrak{O}_2\}$. Let there be just one constant, a_1, and the variable x.[12] Be $i(a_1) = a_1$, $E = \{\langle a_1, a_2\rangle\}$, and $E^- = \varnothing$. Now, consider the quantified formula "$\forall x\, a_1 x$". To evaluate it, we have to consider assignments that map x on a_1 and assignments that map x on a_2. Because of $\langle a_1, a_1\rangle \notin E$, the formula is not true. But because of both $\langle a_1, a_2\rangle \notin E^-$ and $\langle a_1, a_1\rangle \notin E^-$, the formula is not false either. In general, a universally quantified formula $\forall x\, \varphi$ is not true, iff φ is not satisfied by all objects in D^V. And the formula is not false, iff it has no counter-instance, i.e., iff there is no object $a \in D^V$ such that $(Ia/x\varphi) \in E^-$.[13]

One could modify the semantics such that all closed formulas of form-logical systems come out as true or false, but those modifications would have problematic consequences. We will discuss by way of example two possible modifications of the semantics.

(a) The first proposal goes as follows (for φ atomic):

$$\begin{aligned} I_{E,E^-}(\forall x\, \varphi) = \text{TRUE} \quad &:\Leftrightarrow \quad I_{E,E^-}a/x(\varphi) = \text{TRUE for all} \\ &\qquad a \in \{o \in D^V \mid Io/x(\varphi) \in \mathfrak{S}^{\mathfrak{O}}\}. \\ I_{E,E^-}(\forall x\, \varphi) = \text{FALSE} \quad &:\Leftrightarrow \quad I_{E,E^-}a/x(\varphi) = \text{FALSE for at least one} \\ &\qquad a \in \{o \in D^V \mid Io/x(\varphi) \in \mathfrak{S}^{\mathfrak{O}}\}. \end{aligned}$$

The idea is that only legitimate states of affairs are considered in the evaluation of the formula. Problems for this proposal arise when quantifications over molecular formulas are considered. How are we to evaluate the formula $\forall x\, \varphi \wedge \psi$ where x occurs free in both φ and ψ? The obvious proposal is the following (for φ, ψ atomic):

$$\begin{aligned} I_{E,E^-}(\forall x\, \varphi \wedge \psi) = \text{TRUE} \quad :\Leftrightarrow \quad & I_{E,E^-}a/x(\varphi \wedge \psi) = \text{TRUE} \\ & \text{for all } a \in \{o \in D^V \mid Io/x(\varphi) \in \\ & \mathfrak{S}^{\mathfrak{O}}\} \cap \{o \in D^V \mid Io/x(\psi) \in \mathfrak{S}^{\mathfrak{O}}\}. \end{aligned}$$

This proposal has the consequence that the formula $\forall x\, \varphi \wedge \psi$ can be true in a model, where $\forall x\, \varphi$ is false. Just assume that there is an a_0 with $a_0 \in \{o \in D^V \mid Io/x(\varphi) \in \mathfrak{S}^{\mathfrak{O}}\}$ and $a_0 \notin \{o \in D^V \mid Io/x(\psi) \in \mathfrak{S}^{\mathfrak{O}}\}$ and that $Ia_0/x\varphi$ is false.

[12] There are countable infinite further variables, but this need not bother us here.

[13] Open formulas can also receive the third truth-value. Consider the open formula ax and assume that $f^D(i(a))f^D(\beta(x)) \notin \mathfrak{S}^{\mathfrak{O}}$. Then, the interpretation of ax is neither an element of the extension nor of the anti-extension. This could be avoided by demanding that the assignment of the variables respects their form class (just like the interpretation of the constants does): If $x_j \in \mathfrak{T}_k$ then $\beta(x_j) \in \mathfrak{O}_k$, for all $\mathfrak{T}_k \in \mathfrak{T}^*$ and $j \in \mathbb{N}$ (in other words: $\beta[\mathfrak{T}_k] \subseteq \mathfrak{O}_k$, for all k such that there is a $\mathfrak{T}_k \in \mathfrak{T}^*$).

We don't want to buy into this consequence and therefore reject this proposal.

(b) The second proposal is to modify the semantics for quantified formulas as follows:

$$\begin{array}{rcl} I_{E,E^-}(\forall x\, \varphi) = \text{TRUE} & :\Leftrightarrow & I_{E,E^-}\, a/x(\varphi) = \text{TRUE for all } a \in \mathfrak{O}_k \\ & & \text{where } x \in \mathfrak{T}_k. \\ I_{E,E^-}(\forall x\, \varphi) = \text{FALSE} & :\Leftrightarrow & I_{E,E^-}\, a/x(\varphi) = \text{FALSE for at least one} \\ & & a \in \mathfrak{O}_k \text{ where } x \in \mathfrak{T}_k. \end{array}$$

The idea is that only elements of the object-form class that corresponds to the term-form class of the bound variable are considered in the evaluation of the formula. This would correspond to a sortal logic for variables. We shall point to three unwanted consequences of this modification:

(i) The modification allows that the formula $\forall x\, \varphi$ is true in a model, while the formula $\varphi\, a/x$ is false. Just assume that $i(a) \notin f^D(\beta(x))$. Translating this into natural language, it would mean, e.g., that there is a model where "everything is rolling over the floor" is true, but "Russell is rolling over the floor" is false.

(ii) According to the modified semantics, we cannot substitute bound variables *salva veritate*, even if the resulting formulas are well-formed. It can be the case that $\forall x\, \varphi$ is true in a model and $\forall y\, \varphi\, y/x$ is false. Just assume that $f^D(\beta(y)) \neq f^D(\beta(x))$.

(iii) The variable-domain is not, in general, an object-form class itself and, thus there will usually be no term-form class which corresponds to the variable-domain. With the modified semantics, we would not in general have a device for quantifying over the whole variable-domain, but only over subsets of it. This is a severe restriction on the expressivity of the formal language.

These consequences appear problematic enough to motivate investigation of an alternative. We therefore propose a three-valued semantics. Though form-logical systems will usually be three-valued, some turn out to be two-valued. In Section 3, we discuss three different form-logical systems. The most liberal form-logical system (MSL) and the form-logical reconstruction of first-order predicate logic (FPL) turn out to be two-valued, because in these systems all elements of D^V are in the same object-form class and this object-form class corresponds to the only term-form class that contains variables. A third system, illustrating a Wittgensteinian *Begriffsschrift*, will indicate the source for paracompleteness in most form-logical systems.

Completeness. The calculus of the general form logic is a modification of the standard calculus for a three-valued strong Kleene logic. There also exists a proof of the completeness of all form-logical systems. The completeness of the form-logical reconstruction of predicate logic, as of any other form-logical system (see below), is then a consequence of this general result. The strategy of the completeness proof is as follows: we show that for every consistent set X of formulas and every formula φ not deducible from X there is a consistent set of formulas that is deductively closed, contains examples, is strongly disjunctive and does not contain φ as a member. This set can then be used to define a model such that a formula is true in the model if and only if it is a member of the set. It then follows immediately that $X \vDash \varphi$ implies $X \vdash \varphi$. (See the appendix for the proof in full.)

3. Three Form-Logical Systems

We present three specific form-logical systems: (a) We explicate the simplest form-logical system (MSL), in which all terms are classified in the same form class. (b) We show that first-order predicate logic can be translated into a form-logical system (FPL); thus classical first-order predicate logic can be understood as a syntactical specialization of form logic. (c) We illustrate the idea of a Wittgensteinian *Begriffsschrift*, where a sequence of signs is a well-formed atomic proposition iff it corresponds to an atomic possibility. For ease of application, assume that our systems share the following alphabet and structure:

The alphabet:

(i) Variables: x_1, x_2, ...
(ii) Constants: *Russell*, *the egg*, *1*, *is poached*, *is rolling over the floor*, *is (numerically) greater than*, *is fun*.

The structure A is the pair $\langle D, i\rangle$ defined as follows:

$D = \{$Russell, Frege, the egg, the number 1, the property of being poached, the property of rolling over the floor, the relation of being greater than, the property of being fun$\}$.

The function i is defined as follows:

i(*Russell*) = Russell; i(*the egg*) = the egg; i(*1*) = the number 1;
i(*is poached*) = the property of being poached;
i(*is rolling over the floor*) = the property of rolling over the floor;

i(*is greater than*) = the relation of being greater than;
i(*is fun*) = the property of being fun.

Note that the domain contains an unnamed object, namely Frege.

a. The minimally syntactical form logic (MSL)

In order to obtain the most liberal form-logical system, we choose the trivial partition for the set T of all terms; all terms are in the same term-form class $\mathfrak{T}_0 : \mathfrak{T}_0 = T$.

The set of sequences of term-form classes, ${}_{\mathrm{MSL}}\mathfrak{S}^{\mathfrak{T}}$, is the set of all finite, non-singleton sequences of $\mathfrak{T}_0 :_{\mathrm{MSL}} \mathfrak{S}^{\mathfrak{T}} = \mathfrak{T}_0^2 \cup \mathfrak{T}_0^3 \cup \ldots$ (where $\mathfrak{T}_0^i$ is the i-ary Cartesian product of $\mathfrak{T}_0$).

The co-domain D^V of the assignment function is the domain D itself: $D^V = D$. MSL is two-valued, because ${}_{\mathrm{MSL}}D^V = D$ is the only object-form class, $\mathfrak{D}_0$, and $\mathfrak{D}_0$ corresponds to the only term-form class containing variables, namely $\mathfrak{T}_0$.

MSL has minimal syntax rules: all finite, non-singleton sequences of terms are well-formed. Here are some examples of well-formed atomic and quantified formulas of MSL:
Russell is rolling over the floor; the egg is poached; rolling over the floor is fun; Russell is poached; the egg is greater than 1*; is greater than is poached; the number 1 is greater than; Russell Russell*; $\forall x_1(x_1$*is rolling over the floor*); $\forall x_2$(*the egg*x_2); $\exists x_3 \exists x_5 (x_3 x_5)$.

Choose the extension E and the anti-extension E^- as you like (as long as $E \cap E^- = \varnothing$ and $E \cup E^- = \{a_1 \ldots a_n \mid f^D(a_1) \ldots f^D(a_n) \in_{\mathrm{MSL}} \mathfrak{S}^{\mathfrak{D}}\}$. For the sake of definiteness, choose E and E^- as follows:

$E = \{\langle$ the egg, the property of being poached$\rangle$, $\langle$the egg, the property of rolling over the floor$\rangle\}$,

$$E^- = \{a_1 \ldots a_n \mid f^D(a_1) \ldots f^D(a_n) \in_{\mathrm{MSL}} \mathfrak{S}^{\mathfrak{D}}\} \backslash E.$$

With this choice of E and E^-, e.g., the formula *The egg is poached* is true, because of $\langle i$(*the egg*), i(*is poached*)$\rangle \in E$. The formulas *Russell is poached* and *Russell Russell* are false since $\langle i$(*Russell*), i(*is poached*)$\rangle \in E^-$ and $\langle i$(*Russell*), i(*Russell*)$\rangle \in E^-$. The unrestrictive syntax rules have as a consequence that many ‘meaningless’ strings of signs appear as well-formed formulas. Some may perceive this to be a sign of excessive liberalism. We see no problem in this consequence, since — a sensible choice of extension and anti-extension presupposed — the meaningless strings of signs will not come out as true. For many purposes, e.g., the formalisation of ordinary-language arguments and proofs, MSL is as useful as any of its form-logical competitors.

b. The form-logical reconstruction of first-order predicate logic (FPL)

We can construct a form-logical system which exactly matches classical first-order predicate logic. In this system, only a proper subset of the well-formed formulas of MSL is well-formed (assuming the same alphabet).

The constants are divided into exactly $n+1$ term-form classes $\mathfrak{T}_i$ (for some $n \geqslant 1$). All variables are in term-form class $\mathfrak{T}_0$. Intuitively, the form class $\mathfrak{T}_0$ is the form class of the individual constants and the variables, and form class $\mathfrak{T}_i$ (for $1 \leqslant i \leqslant n$) is the form class of the relations of arity i. If we conceive of all form classes $\mathfrak{T}_i$ (for $1 \leqslant i \leqslant n$) as collectively capturing all relations, this classification of terms captures the dualism of terms in predicate logic.

We determine the set of term-form class sequences as follows:

$_{\mathrm{FPL}}\mathfrak{S}^{\mathfrak{T}} = \{\mathfrak{T}_i \mathfrak{T}_0^i \mid i \geqslant 1\}$ (where $\mathfrak{T}_0^i$ is the sequence of i occurrences of $\mathfrak{T}_0$).

The co-domain of the assignment function is $\mathfrak{O}_0$: $_{\mathrm{FPL}}D^V = \mathfrak{O}_0$. FPL is two-valued, because $_{\mathrm{FPL}}D^V = \mathfrak{O}_0$ corresponds to $\mathfrak{T}_0$, which is the only term-form class containing variables. Thus, in FPL the quantifiers range only over the domain of individuals (traditionally conceived).

In FPL, the terms of our example are classified into term-form classes as follows:

$\mathfrak{T}_0 = \{$*Russell, 1, the egg,* $x_1, x_2, \ldots\}$; $\mathfrak{T}_1 = \{$*is poached, is rolling over the floor, is fun*$\}$; $\mathfrak{T}_2 = \{$*is greater than*$\}$. The objects are classified into object-form classes as follows: $\mathfrak{O}_0 = \{$Russell, Frege, the number 1, the egg$\}$; $\mathfrak{O}_1 = \{$the property of being poached, the property of rolling over the floor, the property of being fun$\}$, $\mathfrak{O}_2 = \{$the relation of being greater than$\}$.

Here are some examples of atomic and quantified well-formed formulas of FPL: *Russell is rolling over the floor*; *the egg is poached*; *the egg is greater than 1*; *Russell is poached*; *1 is rolling over the floor*; $\forall x_1(x_1$ *is poached*$)$; $\exists x_2(x_2$ *is rolling over the floor*$)$. Examples of atomic and quantified formulas that are not well-formed are: *Russell Russell*; *being greater than 1 is poached*; *being poached is rolling over the floor*; *rolling over the floor is fun*; $\forall x_2($*the egg* $x_2)$.

FPL is much more restrictive than MSL and thus avoids some "meaningless" strings of signs to come out as well-formed. However, it still allows the arguably meaningless *The egg is greater than 1* and *Russell is poached* as well-formed formulas, while it rules out the arguably meaningful (though certainly false) *Rolling over the floor is fun*.

c. A Wittgensteinian *Begriffsschrift* (W)

An especially interesting form-logical system conforms to the demands of Wittgenstein's *Tractatus*. It constitutes a Wittgensteinian *Begriffsschrift* where all and only those sentences expressing metaphysically possible atomic states of affairs — Wittgenstein's "meaningful" atomic propositions — are treated as well-formed. This can be achieved by a suitable classification of the terms into form classes, such that two terms are in the same form class if and only if they have the same Wittgensteinian logical form, i.e., the same combinatorial possibilities. In such a system first-order possibility is represented already in form-logical syntax. We will now formulate such a modally adequate Wittgensteinian *Begriffsschrift* for the alphabet and the structure introduced above (we assume an intuitive, pre-theoretical understanding of the atomic possibilities).

The terms are divided into term-form classes as follows: *Russell* $\in \mathfrak{T}_1$; *the egg* $\in \mathfrak{T}_2$; *1* $\in \mathfrak{T}_3$; *is poached* $\in \mathfrak{T}_4$; *is rolling over the floor* $\in \mathfrak{T}_5$; *is greater than* $\in \mathfrak{T}_6$; *is fun* $\in \mathfrak{T}_7$. Additionally, let there be countably infinitely many variables in every term-form class.

The objects of D are divided into object-form classes as follows: $\mathfrak{O}_1 =$ {Russell, Frege}, $\mathfrak{O}_2 =$ {the egg}, $\mathfrak{O}_3 =${the number 1}, $\mathfrak{O}_4 =${the property of being poached}, $\mathfrak{O}_5 =$ {the property of rolling over the floor}, $\mathfrak{O}_6 =$ {the relation of being greater than}, $\mathfrak{O}_7 =${the property of being fun}.

We determine the set of term-form class sequences as follows:

$$_{\mathrm{W}}\mathfrak{S}^{\mathfrak{T}} = \{\mathfrak{T}_1\mathfrak{T}_5, \mathfrak{T}_2\mathfrak{T}_4, \mathfrak{T}_2\mathfrak{T}_5, \mathfrak{T}_6\mathfrak{T}_3\mathfrak{T}_3, \mathfrak{T}_5\mathfrak{T}_7\}.$$ [14]

The set of object-form class sequences isthen as follows:

$$_{\mathrm{W}}\mathfrak{S}^{\mathfrak{O}} = \{\mathfrak{O}_1\mathfrak{O}_5, \mathfrak{O}_2\mathfrak{O}_4, \mathfrak{O}_2\mathfrak{O}_5, \mathfrak{O}_6\mathfrak{O}_3\mathfrak{O}_3, \mathfrak{O}_5\mathfrak{O}_7\}.$$

The co-domain of the assignment function is D: $_{\mathrm{W}}D^V = D$.

[14] For the sake of illustration, we ignore that, arguably, Wittgenstein would not have accepted the proposition 'The number 1 is greater than the number 1' as well-formed, since this is a necessary atomic falsehood.

Choose E and E^- as you like (as long as $E \cap E^- = \varnothing$ and $E \cup E^- = \{a_1 \dots a_n \mid f^D(a_1) \dots f^D(a_n) \in {}_{\mathrm{W}}\mathfrak{S}^{\mathfrak{O}}\}$). For the sake of definiteness, choose E and E^- as follows:

$E = \{\langle$ the egg, the property of being poached$\rangle$, $\langle$the egg, the property of rolling over the floor$\rangle\}$,

$E^- = \{\langle$ Russell, the property of rolling over the floor$\rangle$, $\langle$ Frege, the property of rolling over the floor$\rangle$, $\langle$the property of being greater than, the number 1, the number 1$\rangle$, $\langle$the property of rolling over the floor, the property of being fun$\rangle\}$.

The set ${}_{\mathrm{W}}\mathfrak{S}^{\mathfrak{T}}$ is chosen such that exactly those atomic formulas that express possible states of affairs are well-formed. We will illustrate the syntax of W with some examples. We start with formulas without variables, continue with open formulas and finally give some examples of quantified formulas:

(i) Well-formed: *Russell is rolling over the floor*; *the egg is poached*; *the egg is rolling over the floor*; *rolling over the floor is fun.*
(ii) Not well-formed: *The egg is greater than 1*; *Russell is poached*; *1 is rolling over the floor*; *The egg is greater than*; *Russell Russell.*

Because of $E \cup E^- = \{a_1 \dots a_n \mid f^D(a_1) \dots f^D(a_n) \in {}_{\mathrm{W}}\mathfrak{S}^{\mathfrak{O}}\}$, all well-formed atomic formulas without variables are either true or false. To discuss some examples of atomic formulas with variables, let $x_i \in \mathfrak{T}_i$ for $(i \in \{1, \dots, 7\})$. We begin with open formulas.

(i) Well-formed: x_1 *is rolling over the floor*; x_2 *is rolling over the floor* x_2 *is poached*; *the egg* x_4; $x_5 x_7$.
(ii) Not well-formed: x_3 *is rolling over the floor*; x_1 *is poached*; *the egg* x_7; $x_3 x_5$.

Whether a formula with free variables is well-formed depends on the term-form classes of the free variables. Variables of different term-form classes are not, in general, substitutable *salva congruitate.*

The truth-value of an open formula depends on the variable-assignment β, which assigns an arbitrary element of D^V to each variable. If $\beta(x_i) =$ the egg, then the formula x_i *is rolling over the floor* gets assigned the truth-value TRUE. If $\beta(x_i) =$ Frege, then the formula gets assigned the truth-value FALSE. And if $\beta(x_i) =$ the number 1, then the formula gets assign the third truth-value "$*$".

Finally, here are some examples of quantified formulas:

(i) Well-formed:
$\forall x_1(x_1$ *is rolling over the floor*); $\exists x_2(x_2$ *is rolling over the floor*); $\forall x_2(x_2$*is poached*); $\forall x_4$ (*the egg is* x_4); $\exists x_5 \exists x_7(x_5 x_7)$.

(ii) Not well-formed:
$\forall x_3(x_3$ *is rolling over the floor*); $\exists x_1(x_1$ *is poached*); $\forall x_7$ (*the egg* x_7); $\exists x_3 \exists x_5(x_3 x_5)$.

Again, variables of different term-form classes are not, in general, substitutable *salva congruitate*. The truth-values of the quantified formulas are as follows: The formula $\forall x_1(x_1$*is rolling over the floor*) gets assigned the truth-value FALSE, because, e.g., $\langle$Frege, the property of rolling over the floor$\rangle \in E^-$ (and Frege $\in D^V$). The formula $\exists x_2(x_2$ *is rolling over the floor*) gets assigned the truth-value TRUE, because $\langle$the egg, the property of rolling over the floor$\rangle \in E$ (and the egg $\in D^V$). The formula $\forall x_2(x_2$ is poached) gets assigned the truth-value "*". The formula is not false, because there is no object $a \in D^V$ with $\langle a$, the property of being poached$\rangle \in E^-$. The formula is not true either, because, e.g., $\langle$Russell, the property of being poached$\rangle \notin E$. Intuitively speaking, $\forall x\, \varphi$ gets assigned the third truth-value iff (i) all objects in D^V that can possibly satisfy φ, do satisfy φ, and (ii) there is at least one object in D^V that cannot possibly satisfy φ. And $\exists x\, \varphi$ gets assigned the third truth-value iff (i) there is no object in D^V that satisfies φ, and (ii) there is at least one object in D^V that cannot possibly satisfy φ. Given a suitable multiplicity of form classes and no restriction on the domain for the variables, a Wittgensteinian *Begriffsschrift* will always come out as three-valued. It should be noted that three-valuedness is introduced by open and quantified formulas only. Atomic propositions are always either true or false.

4. Conclusion

Form logic does not rest on the dualistic presupposition of PL but rather allows for any number of different types of terms and is thus syntactically more general than PL. This provides the possibility of systems with arbitrary sets of well-formed atomic formulas. MSL is the most liberal system in that it harbours a single term-form class only and allows all sequences of terms as well-formed. PL can be understood (ignoring arity-differences for predicates) as a dualistic specialization of form logic. While form-logical systems are often three-valued, MSL and FPL are two-valued.

Generic form logic does not determine a single system, but presents us with an infinite number of form-logical alternatives. The individual choice will depend, for example, on the purpose the system is to fulfil and on the metaphysical and linguistic background conceptions. A Fregean will opt for FPL, since its dualism may be taken to represent the distinction between complete and incomplete non-logical terms. More generally, a philosopher who holds that language essentially adheres to the subject-predicate form, perhaps because he believes language to reflect the metaphysical dualism between individuals and universals, may equally opt for FPL. Form logic is able to accommodate the convictions of conservative term dualists.

Philosophers of different persuasions, however, might wish to represent their alternative views also in the syntax of the formal language they work with. Ramsey (1925), MacBride (2005) and Freitag (2009) have argued that none of the traditional criteria for drawing the distinction between individuals and universals is adequate. As an alternative to the traditional dualism, one might advocate ontological monism, the view that all sub-propositional entities are of one and the same type. Correspondingly, one might want to propose syntactic monism and hence use MSL rather than FPL as one's logic.

Logico-galactic hitchhikers, again, while also rejecting ontological dualism, favour a view according to which entities, and the corresponding names, fall not into one, but into exactly 42 form classes. They then prefer systems considerably more complex than both FPL and MSL, which are perfectly functioning form-logical systems nevertheless.

Still other philosophers might want to put their form-logical system to modal use and pursue the idea of a Wittgensteinian *Begriffsschrift*. They will construct their systems in such a manner that the respective names are, like the corresponding objects, classified in accordance to their combinatorial possibilities (pre-theoretically so conceived). Then, exactly the contingent atomic propositions are well-formed. This kind of pluralism represents non-iterated modality already in the syntax of the system. Modal facts would not, to invoke Wittgenstein's famous distinction, be *said*, but *shown* in the rules of syntax of the system. As a Wittgensteinian predicate logic à la Hintikka/Wehmeier does without the identity sign and provides syntactical rules to represent identity and difference, a Wittgensteinian *Begriffsschrift* dismisses the box-operator in favor of a syntactical means of representing possibility.[15] (Of course,

[15] Of course, a modal use of form-logical systems is by no means mandatory. One may choose to represent modality in a more traditional way. As modal logic

in light of the fact that such systems can be very complex, one might refrain from the attempt to actually construct a Wittgensteinian *Begriffsschrift* for any particular domain, but still appeal to it as the logical ideal.)[16]

These are only a few linguistic, logical and metaphysical considerations in favour of one and against another system. There might be many more. Our purpose was not to enter, or to decide, the debate concerning the correct logical system. We even wish to remain neutral on the question of whether there is a unique correct system. Our purpose was to present a construction schema, which allows us to formulate first-order logical systems which best fit our preferences — whatever they may turn out to be.

Appendix

In this appendix we provide a calculus for form-logical systems and show their completeness.

1. Calculus

The calculus consists of the following rules:

1. $$\frac{\varphi}{\varphi \vee \psi} \qquad\qquad \frac{\psi}{\varphi \vee \psi}$$

2. $$\frac{\varphi \vee \psi \qquad \begin{matrix}[\varphi] \\ \chi\end{matrix} \qquad \begin{matrix}[\psi] \\ \chi\end{matrix}}{\chi}$$ (occurrences of the bracketed hypothesis leading to the indicated occurrences of χ are cancelled by this rule)

3. $$\frac{\varphi \qquad \neg\varphi}{\psi}$$ (for every well-formed formula ψ)

adds the box-operator to the syntax of predicate logic in order to make modal claims expressible, one might add the box-operator to any form-logical system. Any form-logical system may be the basis of a modal, or, more generally, intensional, logic in the classical sense.

[16] A Wittgensteinian *Begriffsschrift* might represent the ideal system for combinatorialism such as proposed, inspired by Wittgenstein, by Brian Skyrms (1981) and David Armstrong (1989).

4. $$\frac{\forall x(\varphi \vee \psi)}{\varphi \vee \forall x\, \psi} \qquad \text{(for } x \notin \mathrm{fr}(\varphi))$$

5. $$\frac{\forall x\, \varphi}{\varphi\, a/x} \qquad \text{(for } i(a) \in D^V)$$ [17]

6. $$\frac{\varphi a/x}{\forall x\, \varphi}$$ (a does neither appear in φ nor in any uneliminated assumption that led to $\varphi a/x$).

7. $$\frac{\varphi}{\neg\neg\varphi}$$

8. $$\frac{\neg\neg\varphi}{\varphi}$$

9. $$\frac{\neg\varphi \qquad \neg\psi}{\neg(\varphi \vee \psi)}$$

10. $$\frac{\neg(\varphi \vee \psi)}{\neg\varphi} \qquad \frac{\neg(\varphi \vee \psi)}{\neg\psi}$$

11. $$\frac{\neg\forall x\, \varphi \qquad \begin{matrix}[\neg\varphi a/x] \\ \psi\end{matrix}}{\psi}$$ (a does neither appear in φ or ψ nor in any uneliminated assumption that led to ψ)

2. Completeness

In this section we prove the completeness of all form-logical systems. The proof follows that of John T. Kearns for the completeness of a three-valued predicate logic (Kearns 1979), but is adapted to our purposes.

[17] In the rule 5: you get ‘$\varphi a/x$’ by substituting ‘a’ for every ‘x’ in ‘φ’. We here make an explicit reference to the structure, a semantic notion. We could avoid this by distinguishing constants referring to objects in D^V and constants referring to objects in $D \setminus D^V$ already syntactically by choosing different symbols for them, e.g., a_1^V, a_2^V, … and a_1^D, a_2^D, …Instead of the clause ‘$i(a) \in D^V$’, we could then just say ‘for all $a_i^V (i \in \mathbb{N})$’. But we think that the introduction of two types of constants is an unnecessary complication.

i. Preliminary definitions

The entailment and consequence relations are defined as usual: A formula φ is *entailed* by a set of formulas Γ in a form-logical system S ($\Gamma \vDash_S \varphi$) iff in every model of S in which every formula of Γ is true, φ is true, too.

A set of formulas Γ *implies* a formula φ ($\Gamma \vdash_{S,L} \varphi$) in a form-logical system S and a language L iff we can deduce φ from Γ in S and L by a finite number of applications of the rules of the calculus.[18]

A set Γ of formulas is *consistent* iff there is no formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.

A set Γ of formulas is *strongly disjunctive* iff for every formula $(\varphi \vee \psi) \in \Gamma$, it is the case that $\varphi \in \Gamma$ or $\psi \in \Gamma$.

A set Γ of formulas is *instantially sufficient* iff for every existentially quantified formula $\exists x\varphi \in \Gamma$ there is a constant a such that $\varphi a/x \in \Gamma$.

ii. Proof

Let X_0 be a consistent set of formulas of S of the language L^0. In the following, we enlarge the set X_0 to a strongly disjunctive, instantially sufficient, and deductively closed set of formulas Z of the system S.

a. Enlarge X_0 to be strongly disjunctive and deductively closed, get ${}^1Y_0$

We select some enumeration ${}^0\varphi_1, {}^0\varphi_2, \ldots$ of the formulas[19] of L^0. Let Y_0^1 be the closure of X_0 under deducibility in L^0. Y_{m+1}^1 is obtained from Y_m^1 as follows: If ${}^0\varphi_{m+1} \in Y_m^1$ and ${}^0\varphi_{m+1} = (\psi \vee \chi)$ and not $Y_m^1 \vdash \psi$ and not $Y_m^1 \vdash \chi$, then $Y_{m+1}^1 = Y_m^1 \cup \{\psi\}$; otherwise $Y_{m+1}^1 = Y_m^1$.

Then Y_ω^1 is the union of $Y_0^1, Y_1^1, Y_2^1, \ldots$. Then Y_0^2 is the closure of Y_ω^1 under deducibility in L^0. The above construction is repeated to obtain

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18 In the following, we often omit reference to a system S, because we do not change the system throughout the proof. We also omit reference to a language L wherever possible. Let us remark here that form-logical systems are *robust* under enrichments of the language or the domain: Merely enriching the alphabet of a system with new terms does not generate a new system as long as the terms are classified into already existing term-form classes. Analogously, we do not generate a new system by merely enriching the domain with new objects, as long as the new objects are classified into already existing object-form classes.

19 The superscript indicates to which language the enumeration refers: ${}^n\varphi_m$ is the m-th formula of L^n.

$Y_0^2, Y_1^2, Y_2^2, \ldots, Y_\omega^2$. Then Y_0^3 is the closure of Y_ω^2 under deducibility in L^0, etc. ${}^1Y_0$ is the union of Y_0^1, Y_0^2, Y_0^3, ...

b. Enlarge ${}^1Y_0$ to be instantially sufficient and deductively closed, get ${}^1Y_0^1$

We use the new constants a_y^x (with $x, y \in \mathbb{N}$) to enlarge the language. You get L^{n+1} from L^n by adding the constants a_1^{n+1}, a_2^{n+1}, a_3^{n+1},.... L^ω is the union of the languages L^n. (In this paragraph we only need the new constants a_y^1 with $y \in \mathbb{N}$, but the constants for languages of order L^n with $n > 1$ will become relevant when the construction procedure is repeated in paragraph (d).) Let each formula of the form "$\exists x\varphi$" of L^n be associated with a unique constant a_k^{n+1} of L^{n+1} such that $\varphi a_k^{n+1}/x$ is well-formed: if $\exists x\varphi$ is the m-th formula in the enumeration of L^n, let a_m^{n+1} be its *associated constant* and $\varphi a_m^{n+1}/x$ be its *distinguished instance*. In L^1, $\varphi\, a_m^1/x$ is the distinguished instance of $\exists x\varphi$.[20]

${}^1Y_{m+1}$ is obtained from ${}^1Y_m$ as follows: If ${}^0\varphi_{m+1} \in^1 Y_m$ and ${}^0\varphi_{m+1} = \exists x\varphi$, then ${}^1Y_{m+1} =^1 Y_m \cup \{\varphi a_{m+1}^1/x\}$, otherwise ${}^1Y_{m+1} =^1 Y_m$. ${}^1Y_\omega$ is the union of ${}^1Y_0$, ${}^1Y_1$, ${}^1Y_2$, ... And ${}^1Y_0^1$ is the closure of ${}^1Y_\omega$ under deducibility in L^1.

c. Enlarge ${}^1Y_0^1$ to be strongly disjunctive and deductively closed, get X_1

We define M_1 and N_1 as follows:

$M_1 = \{\psi \mid \psi$ is a formula of L^0 and $\psi \notin^1 Y_0^1\}$,

$N_1 = \{\varphi a_m^1/x \mid \varphi a_m^1/x$ is the distinguished instance in L^1 of $\exists x\varphi$ and $\exists x\varphi \in M_1\}$.

Let P_1 be the closure of $M_1 \cup N_1$ under disjunction (i.e., if $\varphi \in P_1$ and $\psi \in P_1$, then $(\varphi \vee \psi) \in P_1$).

${}^1Y_{m+1}^1$ is obtained from ${}^1Y_m^1$ as follows (where ${}^1\varphi_{m+1}$ is the $(m+1)$-th formula of L^1):

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[20] Note that $X \vdash \varphi$ in L^n iff $X \vdash \varphi$ in L^{n+1} (for arbitrary $X \subseteq L^n$ and $\varphi \in L^n$). ($X \vdash \varphi$ in $L^n \Rightarrow X \vdash \varphi$ in L^{n+1}) is trivial. We show that ($X \vdash \varphi$ in $L^{n+1} \Rightarrow X \vdash \varphi$ in L^n). Assume that you have a proof of φ from X in L^{n+1}. There occur only a finite number of new constants in the proof. To generate a proof in L^n from the proof in L^{n+1}, you have to successively substitute the new constants with variables of L^n that do not occur free in any formula of the proof. As φ is in L^n, it is not affected by the substitution.

Because the enhancement of the language does not affect the implication relation in any relevant way, we need not relativize it to a language here.

If ${}^1\varphi_{m+1} \in {}^1Y_m^1$ and ${}^1\varphi_{m+1} = (\psi \vee \chi)$ and not ${}^1Y_m^1 \vdash \psi$ and not ${}^1Y_m^1 \vdash \chi$, then (i) if there is no $\sigma \in P^1$ such that ${}^1Y_m^1 \cup \{\psi\} \vdash \sigma$, then ${}^1Y_{m+1}^1 = {}^1Y_m^1 \cup \{\psi\}$, (ii) if there is a $\sigma \in P_1$ such that ${}^1Y_m^1 \cup \{\psi\} \vdash \sigma$, then ${}^1Y_{m+1}^1 = {}^1Y_m^1 \cup \{\chi\}$; otherwise ${}^1Y_{m+1}^1 = {}^1Y_m^1$. ${}^1Y_\omega^1$ is the union of ${}^1Y_0^1$, ${}^1Y_1^1$, ${}^1Y_2^1$, ... And ${}^1Y_0^2$ is the closure of ${}^1Y_\omega^1$ under deducibility in L^1. This procedure is repeated to obtain ${}^1Y_0^3$, ${}^1Y_0^4$,... etc. The set X_1 is the union of ${}^1Y_0$, ${}^1Y_0^2$, ${}^1Y_0^3$, ...

d. Repeat the whole procedure to obtain X_2, X_3, ... and finally their union Z

The procedure used to obtain X_1 is repeated to obtain X_2, X_3,...: We first enlarge X to be strongly disjunctive and deductively closed and get ${}^2Y_0$ (see step a). Then we enlarge ${}^2Y_0$ to be instantially sufficient and deductively closed and get ${}^2Y_0^1$. To do so, we need the new constants a_1^2, a_2^2, a_3^2, ... of the language L^2 (see step b) and the following definition: We define M_{n+1}, N_{n+1}, P_{n+1}, where

$M_2 = \{\psi \mid \psi$ is a formula of L^1 and $\psi \notin X_n\}$,

$N_2 = \{\varphi\, a_m^2/x \mid \varphi\, a_m^2/x$ is the distinguished instance in L^2 of $\exists x\varphi$ and $\exists x\varphi \in M_2\}$,

and P_2 is the closure of $M_2 \cup N_2$ under disjunction.

In a last step, we enlarge ${}^2Y_0^1$ to be strongly disjunctive and deductively closed, so that we get X_2. Analogously, we obtain X_3, X_4, ... with the following definition:

For each X_n, there are M_{n+1}, N_{n+1}, P_{n+1}, where

$M_{n+1} = \{\psi \mid \psi$ is a formula of L^n and $\psi \notin X_n\}$,

$N_{n+1} = \{\varphi a_m^{n+1}/x \mid \varphi a_m^{n+1}$ is the distinguished instance in L^{n+1} of $\exists x\varphi$ and $\exists x\varphi \in M_{n+1}\}$,

and P_{n+1} is the closure of $M_{n+1} \cup N_{n+1}$ under disjunction. Finally, Z is the union of X_0, X_1, X_2, ...

e. Lemmas

Proofs of the following lemmas are omitted when they are straightforward.

LEMMA 1. *Z is consistent.*

LEMMA 2. *Z is strongly disjunctive.*

LEMMA 3. *Z is instantially sufficient.*

LEMMA 4. *Z is closed under deducibility.*

LEMMA 5. *Let φ be a formula of L such that $X_0 \nvdash \varphi$. Then there is a set Z_φ as above such that $\varphi \notin Z_\varphi$.*

PROOF OF LEMMA 5. We construct Z_φ in the same way as we constructed Z above (see a.–d.) except for the following modifications: (i) We do not define P_1 and P_n as above, but let $P_n = \varphi$, (ii) we modify the construction of Y^n_m (where $n, m \in \mathbb{N}$) so that it parallels that of ${}^1Y^n_m$.

To be shown: $\varphi \notin Z_\varphi$. Proof via induction on the construction of Z_φ.

$X_0 \nvdash \varphi$, therefore $\varphi \notin Y^1_0$. We show that if $Y^1_m \nvdash \varphi$, then $Y^1_{m+1} \nvdash \varphi$: For $Y^1_{m+1} = Y^1_m$: Trivial. $Y^1_{m+1} \neq Y^1_m$: Proof by cases:

(i) $Y^1_{m+1} = Y^1_m \cup \{\psi\}$. From the construction rule that there is no $\sigma \in P_1$ with ${}^1Y^1_m \cup \{\psi\} \vdash \sigma$ and from $\varphi \in P_1$, we get $Y^1_m \cup \{\psi\} \nvdash \varphi$.
(ii) $Y^1_{m+1} = Y^1_m \cup \{\chi\}$. From the construction rule $Y^1_m \cup \{\psi\} \vdash \varphi$ and the assumptions that $(\psi \vee \chi) \in Y^1_m$ and $Y^1_m \nvdash \varphi$, we get $Y^1_{m+1} = Y^1_m \cup \{\chi\} \nvdash \varphi$.

Therefore: $Y^1_\omega \nvdash \varphi$. Hence we get $Y^2_0 \nvdash \varphi$ for the deductive closure Y^2_0 of Y^1_ω.

By exactly analogous reasoning we get $Y^n_0 \nvdash \varphi$ for all $n \in \mathbb{N}$. As the set X_1 is the union of ${}^1Y_0$, ${}^1Y^2_0$, ${}^1Y^3_0$, ..., we get $X_1 \nvdash \varphi$. Since φ is a formula of L, φ cannot become a member of the set we construct in any higher step. Thus we get $Z_\varphi \nvdash \varphi$, and therefore $\varphi \notin Z_\varphi$. $\square$

f. Construction of the term model ${}_SI^Z_{E,E^-}$

We construct a term model ${}_SI^Z_{E,E^-}$ for Z. Let all terms of L^ω be in the domain of ${}_SI^Z_{E,E^-}$. The co-domain of the assignment function of ${}_SI^Z_{E,E^-}$ is the union of all term-form classes of ${}_SI^Z_{E,E^-}$ that contain variables. The interpretation I of ${}_SI^Z_{E,E^-}$ is given as follows:

(i) For constants α: $I(\alpha) = i(a) = \alpha$,
(ii) For variables x: $I(x) = \beta(x) = x$,
(iii) $I(\tau_1, \ldots, \tau_n) = I(\tau_1) \ldots I(\tau_n)$.

The terms are classified in object-form classes such that if $\tau \in \mathfrak{T}_k$, then $I(\tau) \in \mathfrak{O}_k$ in ${}_SI^Z_{E,E^-}$ (for all terms τ and all $\mathfrak{T}_k$ in $\mathfrak{T}^*$). The extension E and the anti-extension E^- of ${}_SI^Z_{E,E^-}$ are defined as follows:

$$E = \{\tau_1 \ldots \tau_n \mid Z \vdash \tau_1 \ldots \tau_n\},$$
$$E^- = \{\tau_1 \ldots \tau_n \mid Z \vdash \neg(\tau_1 \ldots \tau_n)\}.$$

THEOREM 1. *For the term model* ${}_S I^Z_{E,E^-}$ *and any formula* φ *of* L^ω*, we get:*

$$ {}_S I^Z_{E,E^-}(\varphi) = \text{TRUE} \Leftrightarrow \varphi \in Z. $$

PROOF. By induction on the rank of the formulas.
Base case: If $r(\varphi) = 0$, then ${}_S I^Z_{E,E^-}(\varphi) = \text{TRUE} \Leftrightarrow \varphi \in Z$ and ${}_S I^Z_{E,E^-}(\varphi) = \text{FALSE} \Leftrightarrow \neg\varphi \in Z$.

Proof of the basic step: $\varphi = \tau_1 \ldots \tau_n$.

$$ \begin{aligned} & {}_S I^Z_{E,E^-}(\tau_1 \ldots \tau_n) = \text{TRUE} \\ & \Leftrightarrow I(\tau_1 \ldots \tau_n) \in E \\ & \Leftrightarrow \tau_1 \ldots \tau_n \in E \\ & \Leftrightarrow Z \vdash \tau_1 \ldots \tau_n \\ & \Leftrightarrow \tau_1 \ldots \tau_n \in Z \text{ (Z deductively closed)} \end{aligned} $$

$$ \begin{aligned} & {}_S I^Z_{E,E^-}(\tau_1 \ldots \tau_n) = \text{FALSE} \\ & \Leftrightarrow I(\tau_1 \ldots \tau_n) \in E^- \\ & \Leftrightarrow \tau_1 \ldots \tau_n \in E^- \\ & \Leftrightarrow Z \vdash \neg(\tau_1 \ldots \tau_n) \\ & \Leftrightarrow \neg(\tau_1 \ldots \tau_n) \in Z \text{ (Z deductively closed)} \end{aligned} $$

Induction Hypothesis (IH): For every φ in L^ω, if $r(\varphi) = n$, then ${}_S I^Z_{E,E^-}(\varphi) = \text{TRUE} \Leftrightarrow \varphi \in Z$ and ${}_S I^Z_{E,E^-}(\varphi) = \text{FALSE} \Leftrightarrow \neg\varphi \in Z$.

Inductive step: Assume (IH), and show that if $r(\varphi) = n + 1$, then ${}_S I^Z_{E,E^-}(\varphi) = \text{TRUE} \Leftrightarrow \varphi \in Z$.

Proof of the inductive step by cases (assume that $r(\psi) = n$):

i) $\varphi = \neg\psi$:
${}_S I^Z_{E,E^-}(\neg\psi) = \text{TRUE}$
$\Leftrightarrow {}_S I^Z_{E,E^-}(\psi) = \text{FALSE}$
$\Leftrightarrow \neg\psi \in Z$ (IH)

ii) $\varphi = \psi \vee \chi$:
${}_S I^Z_{E,E^-}(\psi \vee \chi) = \text{TRUE}$
$\Leftrightarrow {}_S I^Z_{E,E^-}(\psi) = \text{TRUE}$ or ${}_S I^Z_{E,E^-}(\chi) = \text{TRUE}$
$\Leftrightarrow \psi \in Z$ or $\chi \in Z$ (IH)
$\Leftrightarrow (\psi \vee \chi) \in Z$ (deductively closed and strongly disjunctive)

iii) $\varphi = \exists x\psi$:
${}_S I^Z_{E,E^-}(\exists x\psi) = \text{TRUE}$

$\Leftrightarrow$ there is an a with ${}_S I^{Z}_{E,E^-}(\psi a/x) = \text{TRUE}$ (instantially sufficient)
$\Leftrightarrow$ there is an a with $\psi a/x \in Z$ (IH)
$\Leftrightarrow \exists x\psi \in Z$ (deductively closed and instantially sufficient) $\square$

Theorem 2 (Completeness). *Let X be a set of formulas of L and let φ be a formula of L with $X \vDash \varphi$. Then: $X \vdash \varphi$.*

Proof. Trivial for X inconsistent.

Let X be consistent. We prove that $X \nvdash \varphi$ implies $X \nvDash \varphi$.

If $X \nvdash \varphi$, then there is an expansion Z_φ of X with $\varphi \notin Z_\varphi$ which is consistent, strongly disjunctive and deductively closed (Lemma 5). Let ${}_S I^{Z\varphi}_{E,E^-}$ be the term model for Z_φ as described in (f). By Theorem 1, ${}_S I^{Z\varphi}_{E,E^-} \nvDash \varphi$ (because of $\varphi \notin Z_\varphi$) and ${}_S I^{Z\varphi}_{E,E^-} \vDash Z_\varphi$. But ${}_S I^{Z\varphi}_{E,E^-} \vDash Z_\varphi$ implies ${}_S I^{Z\varphi}_{E,E^-} \vDash X$ (because of $X \subseteq Z_\varphi$). Hence $X \nvDash \varphi$. $\square$

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